

# On Hom type algebras

Yaël Frégier; Aron Gohr

Mathematics Research Unit  
162A, avenue de la faïencerie  
L-1511, Luxembourg  
Grand-Duchy of Luxembourg.

## Abstract

Hom-algebras are generalizations of algebras obtained using a twisting by a linear map. But there is a priori a freedom on where to twist. We enumerate here all the possible choices in the Lie and associative categories and study the relations between the obtained algebras. The associative case is richer since it admits the notion of unit element. We use this fact to find sufficient conditions for hom-associative algebras to be associative and classify the implications between the hom-associative types of unital algebras.

## Introduction.

The present paper investigates variations on the theme of *Hom-algebras*, a topic which has recently received much attention from various researchers [2], [4], [12], [14], [15]. Generally speaking, the notion of Hom-algebra over a certain operad is obtained by twisting in a strategic way the identities for the algebra multiplication implied by the operad in question. For instance, an algebra  $(V, \star)$  together with a linear self-map  $\alpha : V \rightarrow V$  is called *Hom-associative* if it satisfies the identity

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha)$$

which is obtained by replacing *id* with  $\alpha$  in the ordinary associativity condition

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id).$$

The study of Hom-associative algebras originates with work by Hartwig, Larson and Silvestrov in the Lie case [5], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlouf and Silverstrov in [7]. A number of classical constructions have been found to have a Hom-counterpart, see e.g. [1], [6], [8], [9], [10], [11], [12] and [13].

When studying the structure theory of hom-associative algebras, one naturally encounters related algebras which, while not obeying the traditional hom-associative identities, satisfy similar identities [3]. For instance, in the proof of the main result of [3], there naturally appear algebraic structures  $(V, \star, \alpha)$  which satisfy the conditions

$$(\alpha(x) \star y) \star z = x \star (y \star \alpha(z))$$

or

$$\alpha(x \star y) \star z = x \star \alpha(y \star z)$$

for all  $x, y, z \in V$ . This observation suggests that in the context of studying Hom-algebraic structures, there is a natural interest in exploring alternative possibilities on “how to twist” the identities of a given classical algebraic category to obtain a Hom-counterpart.

In this paper, we start a systematic exploration of other possibilities to define Hom-type algebras. We will, for the purposes of the present paper, restrict the scope of this first investigation in several ways. First, we will only consider Hom-twisted versions of the associative and Lie categories. Second, we will only consider symmetric twisted identities and limit the “degree” in terms of occurrences of the twisting map  $\alpha$  in the defining identity of the twisted categories.

The paper is partitioned in two main sections. In the first section we introduce, following the ideas outlined above, new types of Hom-Lie algebras and study their properties in special cases. We give several examples of these new types of Hom-Lie algebras and study their relations among each other and to ordinary Lie algebras. In the second section, we introduce in analogy to this work a similar system of new types of Hom-associative algebras. We point out that in the case of unital algebras, these types of Hom-associativity conditions can be partially ordered by restrictiveness, with the traditional hom-associativity condition ending up on top, i.e. as most restrictive. Finally, we introduce *Hom-monoids* to obtain an easy way to construct counterexamples to possible relations between types of Hom-algebras which do not hold.

These counterexamples prove that our partial ordering of hom-type algebras cannot be improved upon.

We end the introduction by fixing some conventions and notations. In this article,  $k$  will by default be a commutative ring,  $K$  a field. Modules and algebras will by default be understood to be over an arbitrary commutative ring. If  $\alpha : G \rightarrow H$  is a homomorphism of groups (rings, modules, etc.) we will denote by  $Ke(\alpha)$  its kernel and by  $Im(\alpha)$  its image.  $V$  will by default be a  $k$ -module.

## 1 Hom-Lie algebras.

In this part we define types of hom-Lie algebras and give some relations between them.

### 1.1 Definitions.

We start by recalling the original definition following [5].

**Definition 1.1.** *A Hom-Lie algebra is a triple  $(V, [\cdot, \cdot], \alpha)$  consisting of a module  $V$  over a commutative ring  $k$ , a bilinear map  $[\cdot, \cdot] : V \times V \rightarrow V$  and a linear space homomorphism  $\alpha : V \rightarrow V$  satisfying*

$$[x, x] = 0 \tag{1}$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \tag{2}$$

for all  $x, y, z$  in  $V$ , where  $\circlearrowleft_{x,y,z}$  denotes summation over the cyclic permutations on  $x, y, z$ . Explicitly, this means

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] := [\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]].$$

Note that, if  $(G, +, 0)$  is a group with identity 0 and  $\varphi : G \times G \rightarrow G$  is a bi-additive map with  $\varphi(x, x) = 0$ , we automatically have  $\varphi(x, y) + \varphi(y, x) = 0$ . On the other hand, if in  $G$  the equation  $x + x = 0$  always implies  $x = 0$ , as is the case for instance if  $G$  is the additive group of a field of characteristic  $\neq 2$ , then the condition  $\varphi(x, y) + \varphi(y, x) = 0$  implies also  $\varphi(x, x) = 0$  for all  $x \in G$ . Therefore, condition (1) in our definitions corresponds to the usual condition of skew-symmetry of a Lie bracket.

Now if we look at (2), it is natural to ask why we chose to twist by  $\alpha$  in the first argument, and not in the second or third? This question is the first motivation for us to suggest the introduction of two new types,  $I_2$  and  $I_3$ , of Hom-Lie algebras:

**Definition 1.2.** *A Hom-Lie algebra of type  $I_2$  is defined by replacing, in Definition 1.1, equation (2) by*

$$\circlearrowleft_{x,y,z} [x, [\alpha(y), z]] = 0. \quad (3)$$

*If one uses*

$$\circlearrowleft_{x,y,z} [x, [y, \alpha(z)]] = 0 \quad (4)$$

*instead, one gets the definition of a Hom-Lie algebra of type  $I_3$ .*

A Hom-Lie algebra in the usual sense should be referred to as "Hom-Lie algebra of type  $I_1$ ", but we will, most of the time, simply use the term "Hom-Lie algebra", for coherence with the usage in the literature.

**Remark 1.1.** *Of course type  $I_2$  and  $I_3$  are the same by skew symmetry of the bracket. Nevertheless we introduce these two types for pedagogical reasons, since they will appear again in the associative category.*

Now we remark that  $\alpha$  has still two more choices for its dinner: it could be applied to the results of the first or of the second bracket, and give two other types.

**Definition 1.3.** *If one replaces in Definition 1.1, equation (2) by*

$$\circlearrowleft_{x,y,z} [x, \alpha([y, z])] = 0 \quad (5)$$

*one gets the definition of a Hom-Lie algebra of type II.*

*If one uses*

$$\circlearrowleft_{x,y,z} \alpha([x, [y, z]]) = 0, \quad (6)$$

*instead, one gets the definition of a Hom-Lie algebra of type III.*

A trivial example of an algebra of type III is given by considering an arbitrary Lie algebra structure on  $V$  (a Hom-Lie algebra where the twisting

is the identity on  $V$ ), together with an arbitrary linear  $\alpha$ . In the other direction, we see that if  $\alpha$  is any *injective* linear map and  $(V, \alpha)$  is a Hom-Lie algebra of type *III*, then  $V$  must be Lie.

Another example, less trivial, is obtained considering the notion of descending central series  $V^n$ , borrowed from Lie theory:  $V^0 := V, V^1 := [V, V], V^2 := [V, V^1], \dots, V^n := [V, V^{n-1}]$ . Considering an arbitrary  $\alpha$  whose kernel contains  $V^2$  gives the second example. One can also obtain examples where  $\alpha$  does not vanish on  $V^2$ . The following example is an extreme case of this insofar as the kernel of  $\alpha$  is one-dimensional, i.e. of lowest possible dimension:

**Example 1.1.** *Let  $K$  be a field and let  $V := K^3$ . We define a bilinear map  $[\cdot, \cdot] : V \times V \rightarrow V$  by*

$$\left[ \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} \right] := \begin{pmatrix} \lambda_1 \mu_3 - \lambda_3 \mu_1 \\ 0 \\ \lambda_2 \mu_3 - \lambda_3 \mu_2 \end{pmatrix}.$$

*It is clear that this map satisfies  $[v, v] = 0$  for all  $v \in V$ . Also, our bracket does not induce a structure of Lie algebra on  $V$ , since e.g. with  $e_1, e_2, e_3$  the canonical basis vectors of  $V$  we have*

$$\circlearrowleft_{e_1, e_2, e_3} [e_1, [e_2, e_3]] = e_1 \neq 0.$$

*Finally, with  $\alpha : V \rightarrow V$  defined through*

$$\alpha \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} := \begin{pmatrix} \lambda_2 \\ \lambda_3 \\ 0 \end{pmatrix}$$

*we see that  $(V, \alpha, [\cdot, \cdot])$  is Hom-III-Lie by a straightforward calculation. But  $V^2$  has as basis the set  $\{e_1, e_3\}$ , so  $\alpha$  does not vanish on  $V^2$ .*

Let us now do a little science fiction and imagine that  $\alpha$  is in fact a morphism of algebra, i.e satisfies  $[\alpha(x), \alpha(x)] = \alpha([x, y]) \ \forall x, y \in V$ . One could then, using this equality twice, rewrite equation (6) in two ways ((7) and (8)). Hence it is natural to also consider:

**Definition 1.4.** *Hom-Lie algebras of types  $III'$  and  $III''$  are defined by respectively replacing in Definition 1.1, equation (2) by*

$$\circlearrowleft_{x, y, z} [\alpha(x), \alpha([y, z])] = 0, \tag{7}$$

$$\circlearrowleft_{x, y, z} [\alpha(x), [\alpha(y), \alpha(z)]] = 0. \tag{8}$$

One can remark that a Hom-Lie algebra of type  $III''$  is nothing else than a Lie algebra structure on the image of  $\alpha$ .

Similarly, (5) leads to the equation (9) which is quadratic in  $\alpha$ . But once we have opened the Pandora's box, we are forced to also consider the other quadratic expressions in  $\alpha$ , (10) and (11):

**Definition 1.5.** *Hom-Lie algebras of types  $II_1$ ,  $II_2$  and  $II_3$  are defined by respectively replacing in Definition 1.1, equation (2) by*

$$\circlearrowleft_{x,y,z} [x, [\alpha(y), \alpha(z)]] = 0, \quad (9)$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, \alpha(z)]] = 0, \quad (10)$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [\alpha(y), z]] = 0. \quad (11)$$

**Remark 1.2.** *It is easy to see that Remark 1.1 applies mutatis mutandis if one replaces  $I_2$  and  $I_3$  by  $II_2$  and  $II_3$ .*

### On the notations:

We need a notation to distinguish all these types of Hom-Lie algebras.

"Hom<sup>type</sup> – Lie algebra" seems appropriate. For example "Hom <sup>$I_2$</sup>  – Lie algebra" will stand for "Hom-Lie algebra of type  $I_2$ ". By Hom<sup>\*</sup> – Lie we will mean a Hom-algebra of simultaneously all types.

We have chosen to divide these types in three classes  $I$ ,  $II$  and  $III$  accordingly to the degree in  $\alpha$ , i.e. the number of occurrences of  $\alpha$  in the defining equations. We consider the "virtual" degree. In  $\alpha([x, y])$  for example,  $\alpha$  is of virtual degree two even if it appears only once. This is because if  $\alpha$  is a morphism for the bracket, one has  $\alpha([x, y]) = [\alpha(x), \alpha(y)]$  which is really of degree two in  $\alpha$ . We hope that this choice will help the reader to memorize easily the subdivision in classes.

We used the "prime" notation to remember that  $III'$  is derived from  $III$ , and that  $III''$  is derived from  $III'$ . Accordingly,  $II_1$ ,  $II_2$  and  $II_3$  should have been denoted by  $II'_1$ ,  $II'_2$  and  $II'_3$ , but we decided to omit the upper-script prime, since the lower script enables already to distinguish these types.

We have chosen the ordering in the classes  $II$  and  $I$  in a way that they coincide under the symmetry  $S$  which consists of interchanging the role of  $\alpha$  and  $id$  (the identity of  $V$ ), namely for example  $S([Id(x), [\alpha(y), \alpha(z)]]) := [\alpha(x), [Id(y), Id(z)]]$ .

Finally we introduce the Jacobiator associated to each of these structures as the left hand side of the defining equation of the  $Hom^{type} - Lie$  algebra under consideration. One denotes it by  $J_\alpha^{type}$ . As an example,

$$J_\alpha^{II_1} := \odot_{x,y,z} [x, [\alpha(y), \alpha(z)]].$$

In particular  $S(J_\alpha^{I_i}) = J_\alpha^{II_i}$  for  $1 \leq i \leq 3$ .

We conclude this part by the following table which summarizes the list of types of Hom-Lie algebras:

$I_1$	$\odot_{x,y,z} [\alpha(x), [y, z]] = 0$
$I_2$	$\odot_{x,y,z} [x, [\alpha(y), z]] = 0$
$I_3$	$\odot_{x,y,z} [x, [y, \alpha(z)]] = 0$
$II_1$	$\odot_{x,y,z} [x, [\alpha(y), \alpha(z)]] = 0$
$II_2$	$\odot_{x,y,z} [\alpha(x), [y, \alpha(z)]] = 0$
$II_3$	$\odot_{x,y,z} [\alpha(x), [\alpha(y), z]] = 0$
$III$	$\odot_{x,y,z} \alpha([x, [y, z]]) = 0$
$III'$	$\odot_{x,y,z} [\alpha(x), \alpha([y, z])] = 0$
$III''$	$\odot_{x,y,z} [\alpha(x), [\alpha(y), \alpha(z)]] = 0$

## 1.2 Relations among these types.

Now that we have these new types of Hom-Lie algebras, it is natural to seek for relations among them.

**Proposition 1.1.** *Let us suppose that  $(V, [\cdot, \cdot])$  is a Lie algebra and consider  $(V, [\cdot, \cdot], \alpha)$ ,*

1. *if it is a hom-Lie algebra of type  $I_2$ , it is necessarily also of type  $I_1$ ,*
2. *if it is a hom-Lie algebra of type  $II_2$ , it is necessarily of type  $II_1$ .*

In particular

**Corollary 1.1.** *If moreover  $\alpha$  is a morphism of  $[\cdot, \cdot]$ , being of type  $II_2$  implies to be of type  $II_1$  which in turn is equivalent to be of type  $II$ .*

*Proof.* We start by proving the assertion that a hom-Lie algebra of type  $I_2$  is necessarily of type  $I_1$ . Let us first establish the following property:

$$J_\alpha^{I_1} = -J_\alpha^{I_2} - J_\alpha^{I_3}. \quad (12)$$

Indeed, since the bracket satisfies the Jacobi identity, one has:

$$[\alpha(x), [y, z]] = -[z, [\alpha(x), y]] - [y, [z, \alpha(x)]].$$

Summing this relation over cyclic permutations leads to the desired property.

Now let us suppose that we have a  $Hom^{I_2} - Lie$  algebra, i.e.  $J_\alpha^{I_2} = 0$ . Remark 1.2 implies that one also has  $J_\alpha^{I_3} = 0$ . The property (12) enables to conclude.

The proof that a hom-Lie algebra of type  $II_2$  is necessarily of type  $II_1$  is almost the same, one just needs to read the preceding proof after having applied the symmetry  $S$ .  $\square$

The reverse implication in the preceding proposition would need, to hold, that  $J_\alpha^{I_2} + J_\alpha^{I_3} = 0 \Rightarrow J_\alpha^{I_2} = J_\alpha^{I_3} = 0$ . It is natural to ask for a necessary and sufficient condition for this last implication. We do not know the answer to this question but we can remark the "self-adjointness" condition  $[\alpha(x), y] = [x, \alpha(y)] \forall x, y \in V$  on  $\alpha$  is sufficient if the underlying  $k$ -module  $V$  is 2-torsion-free. But this condition of self-adjointness is fairly strong, implying for instance that  $[\alpha(x), x] = 0$  for all  $x \in V$  if  $V$  is a vector space over a field of characteristic  $\neq 2$ , and is therefore in most cases unsatisfied. The following example shows that  $J_\alpha^{I_2} + J_\alpha^{I_3} = 0$  does not indeed imply  $J_\alpha^{I_2} = 0$ , as would be expected:

**Example 1.2.** Let  $K$  be a field with  $\text{char}(K) \neq 2$  and let  $V := K^2$ . We define on  $V$  the bracket

$$\left[ \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \right] := \begin{pmatrix} 0 \\ \lambda_1 \mu_2 - \lambda_2 \mu_1 \end{pmatrix}.$$

It is clear that this is skew-symmetric, and direct calculation verifies the Lie identity. Set now

$$\alpha \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} := \begin{pmatrix} \lambda_1 + \lambda_2 \\ \lambda_2 \end{pmatrix}.$$

Then we see

$$\left[ \alpha \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \left[ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right] \right] = \begin{pmatrix} 0 \\ a_1 b_1 c_2 + a_2 b_1 c_2 - a_1 b_2 c_1 - a_2 c_1 b_2 \end{pmatrix}$$

and summing up cyclic permutations of the term in the second component this implies that condition  $J_\alpha^{I_1}$  is satisfied. On the other hand we have  $[e_1, [\alpha(e_2), e_2]] \neq 0$ , so  $(V, [\cdot, \cdot])$  is not of type  $I_2$ .



Let again  $V$  be an arbitrary  $k$ -module, suppose that  $[\cdot, \cdot] : V \times V \rightarrow V$  is a bilinear map and suppose that  $\alpha : V \rightarrow V$  is linear. We consider now on  $V$  the bracket  $[\cdot, \cdot]_\alpha$  defined by  $[a, b]_\alpha := [a, b] + [\alpha(a), b] + [a, \alpha(b)]$ . We have the following:

**Proposition 1.2.** *Let us suppose that  $(V, [\cdot, \cdot], \alpha)$  is a  $\text{Hom}^*$ -Lie, then  $[\cdot, \cdot]$  is a Lie algebra if and only if  $[\cdot, \cdot]_\alpha$  is a Lie algebra.*

*Proof.* Let us introduce the notation  $\mathfrak{J}_\alpha$ , (resp  $\mathfrak{J}$ ) for the jacobiator of  $[\cdot, \cdot]_\alpha$  (resp  $[\cdot, \cdot]$ ). A direct computation gives that

$$\mathfrak{J}_\alpha = \mathfrak{J} + J_\alpha^{I_1} + J_\alpha^{I_2} + J_\alpha^{I_3} + J_\alpha^{II} + J_\alpha^{II_2} + J_\alpha^{II_3}. \quad (13)$$

Being a  $\text{Hom}^*$ -Lie algebra implies that this last equality reduces to

$$\mathfrak{J}_\alpha = \mathfrak{J},$$

hence the conclusion.  $\square$

**Remark 1.3.** *We did not use the hypothesis of being a  $\text{Hom}$ -Lie algebra of types III, III', III'' and II<sub>1</sub>*

**Proposition 1.3.** *Let us suppose that  $\alpha$  is a morphism for  $[\cdot, \cdot]$  and that  $[\cdot, \cdot]$  is a Lie algebra, then  $[\cdot, \cdot]_\alpha$  is also a Lie algebra.*

*Proof.* Let us consider again the equation

$$(13) \quad \mathfrak{J}_\alpha = \mathfrak{J} + J_\alpha^{I_1} + J_\alpha^{I_2} + J_\alpha^{I_3} + J_\alpha^{II} + J_\alpha^{II_2} + J_\alpha^{II_3}.$$

It becomes, under the use of properties (12) and  $S(12)$ ,

$$\mathfrak{J}_\alpha = \mathfrak{J} + J_\alpha^{II} - J_\alpha^{II_1}.$$

Now the hypothesis of being a morphism implies that  $J_\alpha^{II} = J_\alpha^{II_1}$ , and hence the conclusion.  $\square$

Of course the preceding proposition does not involve  $\text{Hom}$ -algebras in its formulation, but we think it is nevertheless useful to state it because it admits the following generalization:

**Proposition 1.4.** *Let us suppose that  $([\cdot, \cdot], \alpha)$  is simultaneously a  $\text{Hom}^{II}$  and a  $\text{Hom}^{II_1}$ -algebra and that  $[\cdot, \cdot]$  is a Lie algebra, then  $[\cdot, \cdot]_\alpha$  is also a Lie algebra.*

*Proof.* In the preceding proof, the hypothesis of being a morphism was used via the property  $J_\alpha^{II} = J_\alpha^{II_1}$ . But this last property is still satisfied under our new assumption ( $0 = 0!$ ), which ends the proof.  $\square$

It is a good point to make a little pause and have some side remarks. If one compares the two previous statements, one sees that being simultaneously of Hom-algebra types  $\Phi$  and  $\Phi'$  can be thought of as a replacement of the notion morphism. But on the other hand the defect of being a morphism, i.e. the difference  $C_\alpha(a, b) := [\alpha(a), \alpha(b)] - \alpha([a, b])$  is, in geometry, at the heart of the notion of curvature, and has many applications such as characteristic classes for example. This at once suggests to first introduce the defect of being simultaneously  $Hom^\Phi$  and  $Hom^{\Phi'}$ :  $C_\alpha^\Phi(x, y, z) := \odot_{x,y,z} [x, \alpha([y, z])] - [x, [\alpha(y), \alpha(z)]]$  (of course one can similarly define  $C_\alpha^\star$  for the other types). A natural question to investigate, would be to which extent the constructions in homological algebra, based on the usual notion of curvature, can be generalized in this framework. A second question would be to see if these Hom-algebras carry any geometrical meaning, i.e. to search for geometrical properties of tensors of type 1-1 on a manifold such that the Lie algebra of vector fields, with this tensor, form a Hom-algebra.

## 2 Hom-associative algebras.

In this part we start, applying the ideas of the previous part to the associative category, by getting in section 2.1 the list of different types of Hom-associative structures one can consider. Subsequently, we focus our studies on *unital* Hom-associative algebras since much more can be said about this special case than in general. In particular we state a hierarchy on such types of unital hom-algebras, i.e a complete classification of implications between them. We introduce hom-monoids which will give a usefull tool to build counterexamples.

We turn then in section 2.2 to the proof of this hierarchy. We start by proving implications that do hold and then give counter examples to the others.

### 2.1 Types, unitality and hierarchy.

#### Types of Hom-associative algebras.

Hom-associative algebras were introduced in [7], as examples hom-structures. But there are more types of hom-associative algebras than the ones considered in [7]. Analogously to hom-Lie algebras, one gets the following new types of hom-associative algebras:

		<i>II</i>	$x \star \alpha(y \star z) = \alpha(x \star y) \star z$
$I_1$	$\alpha(x) \star (y \star z) = (x \star y) \star \alpha(z)$	$II_1$	$x \star (\alpha(y) \star \alpha(z)) = (\alpha(x) \star \alpha(y)) \star z$
$I_2$	$x \star (\alpha(y) \star z) = (x \star \alpha(y)) \star z$	$II_2$	$\alpha(x) \star (y \star \alpha(z)) = (\alpha(x) \star y) \star \alpha(z)$
$I_3$	$x \star (y \star \alpha(z)) = (\alpha(x) \star y) \star z$	$II_3$	$\alpha(x) \star (\alpha(y) \star z) = (x \star \alpha(y)) \star \alpha(z)$
<i>III</i>	$\alpha(x \star (y \star z)) = \alpha((x \star y) \star z)$		
<i>III'</i>	$\alpha(x) \star \alpha(y \star z) = \alpha(x \star y) \star \alpha(z)$		
<i>III''</i>	$\alpha(x) \star (\alpha(y) \star \alpha(z)) = (\alpha(x) \star \alpha(y)) \star \alpha(z)$		

For precision, we give the following general definition:

**Definition 2.1.** *Let  $V$  be a set together with two binary operations  $+$  :  $V \times V \rightarrow V$  and  $\star$  :  $V \times V \rightarrow V$ , one self-map  $\alpha$  :  $V \rightarrow V$  and a special element  $0 \in V$ . Then  $(V, +, \star, \alpha, 0)$  is called a hom-ring of type  $T$  if*

- *$(V, +, 0)$  is an abelian group*
- *the multiplication is distributive on both sides*

- $\alpha$  is an abelian group homomorphism
- $\alpha$  and  $\star$  satisfy the associativity condition corresponding to type  $T$ .

*Hom-associative algebras over a commutative ring  $k$  are defined analogously by replacing any additivity conditions in the preceding definitions by corresponding conditions of  $k$ -linearity.*

We remark that as in the associative case, a hom-ring may always be viewed as a hom-algebra with  $\mathbb{Z}$  as base ring. Therefore, although we will in the sequel mainly talk about hom-algebras, of course only results that need conditions on properties of the base ring cannot be put in the hom-ring setting.

It seems that not much can be proven *in general* about the relations among the various types of hom-associative algebras just introduced. However, if additional conditions are imposed which restrict the range of algebras under consideration, the theory becomes much richer. One particularly natural condition which can be imposed is existence of a unit element. We will see in section 2.1 that for *unital* hom-associative algebras, the different types we defined can in some sense be partially ordered by increasing generality.

Let us note that some of these types, namely  $I_2, I_3$  and  $II$  already appeared in [3] but not under that name. One can reformulate these results (the meaning of unitality is precised below):

**Proposition 2.1.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_1$ , then it is also of type  $I_2$ .*

**Definition 2.2.** *Let  $(A, \star, \alpha)$  be a hom-associative algebra. Then  $A$  is called left weakly unital if  $\alpha(x) = cx$  for some  $c \in A$ .*

**Lemma 2.1.** *Let  $(A, \star, \alpha, c)$  be a weakly left unital hom-associative algebra with weak left unit  $c \in A$ , bijective  $\alpha$  and let  $\beta := \alpha^{-1}$ , then  $(A, \star, \beta)$  is a hom-associative algebra of type  $I_3$ , it is also of type  $II$ .*

Proposition 2.1 is the transcription of proposition 1.1 of [3], while lemma 2.1 concatenates lemmata 2.1 and 2.4 of [3].

### Unitality.

In the rest of the paper we assume  $(V, \star, 1)$  to be unital. By unitality we mean the usual notion in algebra, i.e the existence of an element 1 in  $V$  such

that  $1 \star x = x \star 1$  for all  $x$  in  $V$ . The notion of an inverse of  $x$  is also meaningful to some extent:  $x^{-1}$  is a left (resp. right) inverse of  $x$  if it satisfies  $x^{-1} \star x = 1$  (resp.  $x \star x^{-1} = 1$ ). We call  $x^{-1}$  an inverse of  $x$  if it is both a left inverse and a right inverse. There are known examples of non associative algebras with elements admitting different right and left inverses. Hence, since *any* bilinear map  $\star : V \times V \rightarrow V$  induces a hom-associative structure on  $V$  if we take  $\alpha = 0$  as twisting homomorphism, there is no reason to assume that the inverse of  $x$  should be unique if defined.

### Hierarchy on types of unital Hom associative algebras.

The hierarchy alluded to in the introduction can for the relations among first and second types be summarized as follows:

**Proposition 2.2.** *Let  $(V, \star, 1)$  be unital, then one has the following relations between the types of  $(V, \star, 1, \alpha)$ :*

- a)  $II_1 \Leftarrow II \Leftrightarrow I_1 \Rightarrow I_3 \Rightarrow \{I_2, II_2, II_3 \text{ and } II_1\}$
- b)  $I_2 \Rightarrow \{II_1 \Leftrightarrow II_3\}$ .

There are no relations among {types III} and from {types III} to {types I and II}. The relations from {types I and II} to {types III} are given by:

**Proposition 2.3.** *Let  $(V, \star, 1)$  be unital, then one has the following relations between the types of  $(V, \star, 1, \alpha)$ :*

- a)  $I_1 \Leftarrow III, III', III''$
- b)  $I_3 \Rightarrow III''$
- c)  $I_2 \Rightarrow III''$
- d)  $II_2 \Rightarrow III''$
- e)  $II_1, II_3 \Rightarrow III''$

The equivalence of types  $I_1$  and  $II$  will be given in Proposition 2.4. Proposition 2.5 gives that type  $I_1$  implies type  $I_3$  and that type  $II$  implies type  $II_1$ . Proposition 2.6 states that  $I_3$  implies  $I_2, II_2, II_3$  and  $II_1$ . We prove in Proposition 2.7 the exotic implication of Proposition 2.2 b).

Proposition 2.3 is the conjunction of Propositions 2.8, 2.9 and 2.10.

### Hom-monoids

We will now introduce *hom-monoids* and give a short discussion of their relation to hom-algebras. The main motivation is that hom-monoids will

be useful in the construction of counterexamples to relations between the different types of hom-algebras.

**Definition 2.3.** A hom-monoid of type  $I$  is a set  $S$  together with a binary operation  $\star : S \times S \rightarrow S$ , a special element  $1 \in S$  and a map  $\alpha : S \rightarrow S$  such that the following axioms are fulfilled:

$$\begin{aligned} 1 \star x &= x \star 1 = x \\ \alpha(x) \star (y \star z) &= (x \star y) \star \alpha(z). \end{aligned}$$

Similarly, we introduce for each type of hom-associative algebra defined previously the corresponding type of hom-monoid. If we do not specify a type, Type  $I$  will by default be implied.

The following example is clear:

**Example 2.1.** Let  $(V, \star, +, \alpha, 1)$  be a hom-algebra of type  $T$ . Then the multiplicative structure  $(V, \star, \alpha, 1)$  is a hom-monoid also of type  $T$ .

In the other direction, one has the following remark:

**Remark 2.1.** Let  $k$  be a commutative ring and let  $(S, \tilde{\star}, \tilde{\alpha}, 1)$  be a hom-monoid of type  $T$ . Let then  $V$  be the free  $k$ -module over  $S$  and define  $\alpha : V \rightarrow V$  and  $\star : V \times V \rightarrow V$  by linear extension of  $\tilde{\alpha} : S \rightarrow S$  respectively  $\tilde{\star} : S \times S \rightarrow S$  to  $V$ . Then  $(V, \star, \alpha, 1)$  is a unital hom-associative algebra of type  $T$ . We denote the hom-algebra so constructed from a hom-monoid  $S$  by  $k[S]$ .

*Proof.* By construction,  $\alpha$  is linear and  $\star$  is bilinear. Using the distributive laws, one verifies easily that type  $T$  hom-associativity of  $(V, \star, \alpha)$  follows from the corresponding property on the generating set  $S$ . Unitality of  $V$  is clear. This concludes the proof.  $\square$

With respect to exploring the relations between different types of hom-associative algebras, the preceding remark and example show that if type  $T_1$  subsumes type  $T_2$  in the context of unital hom-algebras, then the same holds in the context of hom-monoids and vice versa.

To obtain from hom-monoids examples of hom-algebras which can be written down in a particularly concise way, it will be useful to also set the following definition:

**Definition 2.4.** A hom-monoid  $(S, \star, \alpha, 1)$  of type  $T$  is called a hom-monoid of type  $T$  with zero if there is an element  $0 \in S$ ,  $0 \neq 1$ , such that  $0 \star x = x \star 0 = 0$  for all  $x \in S$  and  $\alpha(0) = 0$ .

Since as with unital hom-algebras of the original type  $I_1$ , the twisting map  $\alpha$  is also in a hom-monoid of type  $I_1$  automatically multiplication with some element inside the structure, one could drop the condition  $\alpha(0) = 0$  in this case from the definition of a hom-monoid with zero. However, for most of the other types it is necessary to impose it separately.

In any case, if  $V$  is a hom-algebra over some commutative ring  $k$  of type  $T$  constructed from a hom-monoid  $S$  with zero of the same type, and if  $0_S$  denotes the zero element in  $S$ , then the submodule  $I := k \cdot 0_S$  becomes a hom-ideal of  $k[S]$ . One can kill this submodule by passing to the appropriate factor algebra  $A := k[S]/I$ . We note that  $A$  still contains a copy of  $S$ . In particular, if  $S$  was (not) of type  $T$ , then  $A$  will be (not) of type  $T$ .

## 2.2 Proof of the hierarchy

The rest of this paper is devoted to the exploration of the logical relationships between the different types of unital hom-algebras. We start by establishing relations that do hold. Counter examples to the other relations are given at the end of this part.

### Equivalence between types $I_1$ and $II$

We start by a lemma which contains the main basic properties allowing computations with Types  $I_1$  and  $II$ . This lemma was proved in [3] (lemma 1.1) under the assumption of being of type  $I_1$ . We recall it and extend it under the assumption of being of type  $II$ .

**Lemma 2.2.** Let  $(V, \star, \alpha, 1)$  be of type  $I_1$  or  $II$ . One has for all  $x, y$  in  $V$ :

$$\ddot{a}) \quad \alpha(x) \star y = x \star \alpha(y),$$

$$\ddot{e}) \quad x \star \alpha(1) = \alpha(x),$$

$$\ddot{i}) \quad \alpha(x \star y) = x \star \alpha(y).$$

*Proof.* The proof of the two first points, assuming the Hom-algebra to be of type  $II$  works along exactly the same lines as the proof of lemma 1.1 in [3] and is left to the reader. To prove i), simply apply the definition of Hom algebra of type  $II$  to the triple  $(x, y, 1)$ :  $\alpha(x \star y) \star 1 \stackrel{II}{=} x \star \alpha(y \star 1)$ .  $\square$

The desired equivalence is then a simple corollary:

**Proposition 2.4.** *Hom-associative algebras of types  $I_1$  and  $II$  are equivalent.*

*Proof.* The contemplation of the following square gives the proof.

$$\begin{array}{ccc} x \star \alpha(y \star z) & \stackrel{II}{=} & \alpha(x \star y) \star z \\ \text{\textcircled{a}} \parallel & & \text{\textcircled{a}} \parallel \\ \alpha(x) \star (y \star z) & \stackrel{I_1}{=} & (x \star y) \star \alpha(z). \end{array}$$

$\square$

### Type implications from $I_1$

Now, we concern ourselves with the types subsumed by the equivalent types  $I_1$  and  $II$ .

**Proposition 2.5.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_1$ , then*

- a) it is also of type  $I_3$ ;*
- b) it is also of type  $II_1$ .*

The proof of proposition 2.5 requires the following equalities taken from proposition 1.1 of [3].

**Lemma 2.3.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_1$ . One has for all  $x, y$  and  $z$  in  $V$ :*

- 1)  $\alpha(x) \star (y \star z) = (\alpha(x) \star y) \star z$ ,*
- 2)  $x \star (y \star \alpha(z)) = (x \star y) \star \alpha(z)$*

*Proof.* The proof of a) comes from contemplation of the following square:

$$\begin{array}{ccc} \alpha(x) \star (y \star z) & \stackrel{I_1}{=} & (x \star y) \star \alpha(z) \\ \text{2.3 1) } \parallel & & \text{2.3 2) } \parallel \\ (\alpha(x) \star y) \star z & \stackrel{I_3}{=} & x \star (y \star \alpha(z)). \end{array}$$



There exists a direct proof of b), but it can also be seen from the chain  $I_1 \xRightarrow{a)} I_3 \xRightarrow{2.6} \{I_2, II_3\} \xRightarrow{2.7} II_1$  which will be proven in the following two sections.  $\square$

### Type implications from $I_3$ .

The main result of this section is

**Proposition 2.6.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_3$ , then*

- a) it is also of type  $I_2$ ;*
- b) it is also of type  $II_2$ ;*
- c) it is also of type  $II_3$ .*

Its proof is based on the following two basic properties:

**Lemma 2.4.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_3$ , then  $\forall x, y \in V$ ,*

- $\alpha$ )  $\alpha(x) = x \star \alpha(1)$ ;*
- $\beta$ )  $x \star \alpha(y) = \alpha(x) \star y$ .*

*Proof of the lemma.* One proves  $\alpha$ ) by applying the definition of  $Hom^{I_3}$ -associativity to the triple  $(x, 1, 1)$ :

$$x \star \alpha(1) = x \star (1 \star \alpha(1)) \stackrel{I_3}{=} (\alpha(x) \star 1) \star 1 = \alpha(x).$$

The proof of  $\beta$ ) is obtained by considering the triple  $(x, y, 1)$  by the use of a):

$$x \star \alpha(y) \stackrel{a)}{=} x \star (y \star \alpha(1)) \stackrel{I_3}{=} (\alpha(x) \star y) \star 1 = \alpha(x) \star y.$$

$\square$

One should resist the temptation to deduce lemma 2.2 from proposition 2.5 and lemma 2.4 since proposition 2.5 relies itself on lemma 2.2. We now turn on to the proof of Proposition 2.6:

*Proof.* We show each statement in turn:

a)

$$\begin{array}{ccc} x \star (\alpha(y) \star z) & \stackrel{I_2}{=} & (x \star \alpha(y)) \star z \\ \beta) \parallel & & \parallel \beta) \\ x \star (y \star \alpha(z)) & \stackrel{I_3}{=} & (\alpha(x) \star y) \star z. \end{array}$$

b)

$$\begin{array}{ccc} (\alpha(x) \star y) \star \alpha(z) & \stackrel{II_2}{=} & \alpha(x) \star (y \star \alpha(z)) \\ I_3 \parallel & & \parallel I_3 \\ x \star (y \star \alpha(\alpha(z))) & & (\alpha(\alpha(x)) \star y) \star z. \\ \beta) \parallel & & \parallel \beta) \\ x \star (\alpha(y) \star \alpha(z)) & \stackrel{I_3}{=} & (\alpha(x) \star \alpha(y)) \star z. \end{array}$$

c)

$$\begin{array}{ccc} \alpha(x) \star (\alpha(y) \star z) & \stackrel{II_3}{=} & (x \star \alpha(y)) \star \alpha(z) \\ \beta) \parallel & & \parallel \beta) \\ \alpha(x) \star (y \star \alpha(z)) & & (\alpha(x) \star y) \star \alpha(z). \\ I_3 \parallel & & \parallel I_3 \\ (\alpha(\alpha(x)) \star y) \star z & & x \star (y \star \alpha(\alpha(z))). \\ \beta) \parallel & & \parallel \beta) \\ (\alpha(x) \star \alpha(y)) \star z & \stackrel{I_3}{=} & x \star (\alpha(y) \star \alpha(z)). \end{array}$$

□

### Exotic implications.

We call these implications exotic since, contrary to the previous ones, they involve two types of hom-algebras in the assumptions.

**Proposition 2.7.** *Let  $(V, \star, \alpha, 1)$  be a unital Hom-associative algebra of type  $I_2$ , then it is of type  $II_3$  if and only if it is of type  $II_1$ .*

*Proof.* The following calculation shows our claim:

$$\begin{array}{ccc} x \star (\alpha(y) \star \alpha(z)) & \stackrel{II_1}{=} & (\alpha(x) \star \alpha(y)) \star z \\ I_2 \parallel & & \parallel I_2 \\ (x \star \alpha(y)) \star \alpha(z) & \stackrel{II_3}{=} & \alpha(x) \star (\alpha(y) \star z). \end{array}$$

□

### Implications from types of families I and II to types of family III.

There are not many relations between these types, except for  $III''$  which is weaker than almost all the other types and  $I_1$  which is stronger than all the other types.

**Proposition 2.8.** *One has the following implications*

- a) *a hom-algebra of type  $I_1$  is necessarily of type  $III$ ,*
- b) *a hom-algebra of type  $I_1$  is necessarily of type  $III'$ ,*
- c) *a hom-algebra of type  $I_1$  is necessarily of type  $III''$ .*

*Proof.* a) is Proposition 2.1, (4).

b)

$$\begin{aligned} \alpha(x) \star \alpha(y \star z) &= \alpha(x \star y) \star \alpha(z) \\ \text{lemma 2.2} \parallel &\quad \parallel \text{lemma 2.2} \\ \alpha(x) \star (\alpha(y) \star z) &\stackrel{I_1}{=} (x \star \alpha(y)) \star \alpha(z). \end{aligned}$$

*Hom  $III'$ -associativity* in c) is obtained applying proposition 2.1 1) to the triple  $\{x, \alpha(y), \alpha(z)\}$ .

□

The last point of the previous proposition can be refined (and implied) by:

**Proposition 2.9.** *A hom-algebra of type  $I_3$  is necessarily of type  $III''$ .*

*Proof.* By Proposition 2.6 a),  $I_3 \Rightarrow I_2$ . But by Proposition 2.10 a) below,  $I_2 \Rightarrow III'$ . □

There are no other implications in this direction from  $I_3$ , as shown by the following counterexample:

$$I_3 \not\Rightarrow III, III' : (e_2 \cdot e_2) := e_1, (e_3 \cdot e_3) := e_3; \alpha(e_1) = \alpha(e_3) = e_3.$$

In particular, since  $I_3$  implies  $\{II_1, II_2, II_3, I_2\}$ , the previous counter examples are also counter examples to  $\{II_1, II_2, II_3, I_2\} \Rightarrow III$  and  $\{II_1, II_2, II_3, I_2\} \Rightarrow III'$ . But can  $III''$  be implied by one of these types?

**Proposition 2.10.** *One has the following implications:*

- a) *a hom-algebra of type  $I_2$  is necessarily of type  $III''$*
- b) *a hom-algebra of type  $II_2$  is necessarily of type  $III''$*
- c) *a hom-algebra of types  $II_1$  and  $II_3$  is necessarily of type  $III''$ .*

The hard point to prove is c), and we will need for it the following

**Lemma 2.5.** *For a hom-algebra  $V$  which is of types  $II_1$  and  $II_3$ , one has  $\forall x, y \in V$  :*

- a)  $\alpha(1) \star \alpha(x) = \alpha(x) \star \alpha(1)$
- b)  $\alpha(\alpha(x)) \star \alpha(1) = \alpha(x) \star \alpha(\alpha(1))$
- c)  $\alpha(x) \star (\alpha(1) \star \alpha(y)) = (\alpha(x) \star \alpha(1)) \star \alpha(y)$
- d)  $\alpha(\alpha(x)) \star \alpha(y) = \alpha(x) \star \alpha(\alpha(y))$ .

*Proof of Lemma 2.5.* We start with (a):

$$\begin{aligned} \alpha(1) \star \alpha(x) &= \alpha(x) \star \alpha(1) \\ \parallel &\parallel \\ (\alpha(1) \star \alpha(x)) \star 1 &\stackrel{II_1}{=} 1 \star (\alpha(x) \star \alpha(1)) \end{aligned}$$

b)

$$\begin{aligned} \alpha(\alpha(x)) \star \alpha(1) &= 1 \star (\alpha(x) \star \alpha(\alpha(1))) \\ \parallel &\parallel II_1 \\ (\alpha(\alpha(x)) \star \alpha(1)) \star 1 &= (\alpha(1) \star \alpha(x)) \star \alpha(1) \\ II_1 \parallel &\parallel a) \\ \alpha(x) \star (\alpha(1) \star \alpha(1)) &\stackrel{II_3}{=} (\alpha(x) \star \alpha(1)) \star \alpha(1) \end{aligned}$$

c)

$$\begin{aligned} \alpha(x) \star (\alpha(1) \star \alpha(y)) &= (\alpha(x) \star \alpha(1)) \star \alpha(y) \\ II_1 \parallel &\parallel II_1 \\ (\alpha(\alpha(x)) \star \alpha(1)) \star y &= x \star (\alpha(1) \star (\alpha(\alpha(y)))) \\ b) \parallel &\parallel b) \\ (\alpha(x) \star \alpha(\alpha(1))) \star y &\stackrel{II_1}{=} x \star (\alpha(\alpha(1)) \star (\alpha(y))) \end{aligned}$$

d)

$$\begin{aligned} \alpha(\alpha(x)) \star \alpha(y) &= \alpha(x) \star \alpha(\alpha(y)) \\ \parallel &\parallel \\ (\alpha(\alpha(x)) \star \alpha(y)) \star 1 &= 1 \star (\alpha(x) \star \alpha(\alpha(y))) \\ II_1 \parallel &\parallel II_1 \\ \alpha(x) \star (\alpha(y) \star \alpha(1)) &= (\alpha(1) \star \alpha(x)) \star \alpha(y) \\ a) \parallel &\parallel a) \\ \alpha(x) \star (\alpha(1) \star \alpha(y)) &\stackrel{c)}{=} (\alpha(x) \star \alpha(1)) \star \alpha(y) \end{aligned}$$

□

*Proof of Proposition 2.10.* One proves a) by applying the definition of type  $I_2$  to the triple  $\{\alpha(x), y, \alpha(z)\}$ , and one proves b) by applying the definition of type  $II_2$  to the triple  $\{x, \alpha(y), z\}$ . The proof of c) comes from contemplation of the following diagram:

$$\begin{array}{ccc}
(\alpha(x) \star \alpha(y)) \star \alpha(z) & \stackrel{III''}{=} & \alpha(x) \star (\alpha(y) \star \alpha(z)) \\
II_1 \parallel & & \parallel II_1 \\
x \star (\alpha(y) \star \alpha(\alpha(z))) & & (\alpha(x) \star \alpha(y)) \star \alpha(z) \\
\text{lem 2.5} \parallel & & \parallel \text{lem 2.5} \\
x \star (\alpha(\alpha(y)) \star \alpha(z)) & \stackrel{II_3}{=} & (\alpha(x) \star \alpha(\alpha(y))) \star z.
\end{array}$$

□

### Counterexamples to intertype relations.

We give now a list of counterexamples to inter-type relations which do *not* hold. All of these counterexamples are constructed by use of the technique of hom-monoids discussed in section 2.1 above. For each example, we first describe what the example is supposed to show. Here,  $T_1, T_2, \dots, T_n \not\Rightarrow T$  means that a hom-monoid which is of simultaneously of types  $Type_i$  is not necessarily also type  $T$ . This is followed by relations between elements of a counterexample hom-monoid of the requisite types. The elements of the hom-monoid structures in question are denoted  $e_1, \dots, e_n$ , where  $e_1$  is supposed the unit element. All hom-monoid structures are with zero, but the zero element is outside the set  $\{e_1, \dots, e_n\}$ . When we do not give a product  $e_i e_j$ , this means that  $e_i e_j = 0$ , except when  $i = 1$  or  $j = 1$ , in which case the product is prescribed by the requirement that  $e_1$  be the unit of the hom-monoid. The hom-monoids in question are not supposed to have elements  $e_i$  with higher index than those appearing in the relations we supply. Likewise, we give only nonzero values of  $\alpha$ . Note that the first three examples use the fact that not every associative structure is also hom-associative of all types.

1.  $I_2 \not\Rightarrow I_3$ :  $\alpha(e_2) = e_1$ .
2.  $I_2 II_2 \not\Rightarrow II_3$ :  $\alpha(e_1) = e_1$ .
3.  $II_1 III_2 II_3 I_2 \not\Rightarrow I_3$ :  $\alpha(e_2) = e_2$ .
4.  $I_3 \not\Rightarrow I_1$ :  $(e_2 \cdot e_2) := e_1$ ;  $\alpha(e_2) = \alpha(e_3) = e_3$ .

5.  $I_2 \not\Rightarrow II_2$ :  $(e_2 \cdot e_2) := e_2, (e_2 \cdot e_3) := e_2; \alpha(e_1) = \alpha(e_2) = e_3$ .
6.  $II_2 \not\Rightarrow I_2$ :  $(e_2 \cdot e_3) = e_3; \alpha(e_1) = e_2$ .
7.  $II_1 II_2 \not\Rightarrow II_3$ :  $(e_2 \cdot e_3) = (e_3 \cdot e_2) = e_1; \alpha(e_1) = e_3$ .
8.  $II_1 \not\Rightarrow II_2$ :  $(e_2 \cdot e_3) = e_1; \alpha(e_1) = e_2$ .
9.  $II_1 II_2 II_3 \not\Rightarrow I_2$ :  $(e_2 \cdot e_3) = e_1, (e_3 \cdot e_2) = e_1; \alpha(e_2) = e_3$ .
10.  $II_2 II_3 \not\Rightarrow II_1$ :  $(e_2 \cdot e_2) = e_1, (e_3 \cdot e_3) = e_2; \alpha(e_1) = e_3$ .
11.  $I_2 II_1 II_3 \not\Rightarrow II_2$ :  $(e_3 \cdot e_2) = e_4, (e_4 \cdot e_3) = e_2; \alpha(e_1) = e_3$ . The next three examples show the absence of relations between Hom types  $III, III'$  and  $III''$ :
12.  $III, III'' \not\Rightarrow III'$ :  $\alpha(e_2) = e_1$ .
13.  $III, III' \not\Rightarrow III''$ :  $(e_2 \cdot e_2) := e_3, (e_3 \cdot e_2) := e_2; \alpha(e_1) = e_2$ .
14.  $III', III'' \not\Rightarrow III$ :  $(e_2 \cdot e_2) := e_3, (e_3 \cdot e_2) := e_3; \alpha(e_3) = e_3$ .  
Lastly, we give an example that shows that the order three types do not subsume any of the other types even when taken together:
15.  $III, III', III'' \not\Rightarrow I_2, II_1, II_2, II_3$ :  $e_2 \cdot e_2 = e_1, e_2 \cdot e_3 = e_1, e_3 \cdot e_2 = e_2, e_3 \cdot e_3 = e_1$  and  $\alpha(e_1) = \alpha(e_2) = \alpha(e_3) = e_3$ .

**Acknowledgements.** The authors were supported in their work by research grant R1F105L15 of the University of Luxembourg (Martin Schlichenmaier). This work was aided by use of the first order logic theorem prover Prover9 and the finite countermodel searcher Mace4. In particular, Mace4 was very useful in the practical application of the Hom-monoid technique for counterexample building in the proof of the “negative” parts of the Hom-associative type hierarchy. Also some of the examples in the Lie case are due to Mace4. Prover9 was useful for quick verification of many parts of the associative hierarchy, although we mostly did not use the proofs offered.

The first author wants to thank the Max Planck Institute for Mathematics in Bonn for excellent working conditions.

We also thank Abdenacer Makhoulouf and Donald Yau for usefull comments on a preliminary draft. This work has been published thanks to the financial support of the ”Fonds National de la Recherche” of Luxembourg.

## References

- [1] H. Ataguema, A. Makhlouf, and S. Silvestrov, *Generalization of  $n$ -ary Nambu algebras and beyond* J. Math. Phys. 50, 083501 (2009). DOI:10.1063/1.3167801, (Preprint version arXiv:0812.4058, 2008).
- [2] Caenepeel, S., Goyvaerts, I., *Hom-Hopf algebras*, (Preprint, arXiv: 0907.0187, 2009)
- [3] Fregier, Y., Gohr, A., *On unitality conditions for Hom-associative algebras*, (Preprint, arXiv:0904.4874, 2009).
- [4] Gohr, A., *On hom-algebras with surjective twisting*, (Preprint, arXiv:0906.3270, 2009).
- [5] Hartwig, J. T., Larsson, D., Silvestrov, S.D., *Deformations of Lie algebras using  $\sigma$ -derivations*, J. Algebra **295** (2006), 314-361.
- [6] Larsson, D., Silvestrov, S. D., *Quasi-hom-Lie algebras, Central Extensions and 2-cocycle-like identities*, J. Algebra **288** (2005), 321-344.
- [7] Makhlouf A., Silvestrov S. D., *Hom-algebra structures*, J. Gen. Lie Theory Appl. Vol **2** (2) 2008, pp 51-64.
- [8] Makhlouf A. and Silvestrov S. D., *Hom-Algebras and Hom-Coalgebras*, Preprints in Mathematical Sciences, Lund University, Centre for Mathematical Sciences, Centrum Scientiarum Mathematicarum, (2008) ; (Preprint, arXiv:0811.0400, 2007).
- [9] Makhlouf A. and Silvestrov S. D., *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, Published as Chapter 17, pp 189-206, S. Silvestrov, E. Paal, V. Abramov, A. Stolin, (Eds.), Generalized Lie theory in Mathematics, Physics and Beyond, Springer-Verlag, Berlin, Heidelberg, (2008).
- [10] Makhlouf A. and Silvestrov S. D., *Notes on formal deformations of Hom-Associative and Hom-Lie algebras*, To appear in Forum Mathematicum.
- [11] Yau D., *Enveloping algebra of Hom-Lie algebras*, J. Gen. Lie Theory Appl. Vol **2** (2) 2008, pp 95-108.
- [12] Yau D., *Hom-algebras and homology*, (Preprint, arXiv:0712.3515, 2007)

- [13] Yau D., *Hom-bialgebras and comodule algebras*, (Preprint, arXiv:0810.4866v1, 2008).
- [14] Yau D., *Hom-quantum groups I: quasi-triangular Hom-bialgebras* (Preprint, arXiv: 0906.4128, 2009).
- [15] Yau D., *Hom-quantum groups II: cobraided Hom-bialgebras and Hom-quantum geometry* (Preprint, arXiv: 0907.1880, 2009).